

* The closed graph theorem:

Closed linear Transformation: Def: Let N and N' be normed linear spaces and let D be a subspace of N . Then a linear transformation $T: D \rightarrow N'$ is said to be closed iff $x_n \in D$, $x_n \rightarrow x$ and $T(x_n) \rightarrow y$ imply $x \in D$ and $y = T(x)$.

Statement:- Let B and B' be Banach spaces and let T be a linear transformation of B onto B' . Then T is a continuous mapping if and only if its graph is closed.

In other words - if X and Y are Banach spaces and if T is a linear operator of X into Y , then T is continuous iff its graph G_T is closed.

Proof:- The 'only if' part, let T be continuous and let T_G be the graph of T . We shall show that $\bar{T}_G = T_G$ and this will prove that T_G is closed.

Since $T_G \subset \bar{T}_G$ always, we need only prove $\bar{T}_G \subset T_G$.

So let $(x, y) \in \bar{T}_G$.

Then (x, y) is an adherent point of T_G .

Hence there exists a sequence $\langle x_n, T(x_n) \rangle$ in T_G such that $(x_n, T(x_n)) \rightarrow (x, y)$ which implies that $x_n \rightarrow x$ and $T(x_n) \rightarrow y$. But, since T is continuous, $x_n \rightarrow x$ implies $T(x_n) \rightarrow T(x)$ and so $y = T(x)$. This shows that $(x, y) = (x, T(x)) \in T_G$ and so $\bar{T}_G \subset T_G$.

The 'if' part. Let T_G is closed. We denote by B_1 the linear space B renormed by

$$\|x\|_1 = \|x\| + \|T(x)\|.$$

$$\text{Now } \|T(x)\| \leq \|x\| + \|T(x)\| = \|x\|_1.$$

Which shows that T is bounded and consequently continuous. It therefore suffices to show that B_1 and B have the same topology, that is, they are homeomorphic. Consider the identity map $I: B_1 \rightarrow B: I(x) = x \forall x \in B_1$.

Clearly I is one-one onto. Further

$$\|I(x)\| = \|x\| \leq \|x\|_1 = \|x\| + \|T(x)\|,$$

which shows that I is bounded and hence continuous.

If we can show that B_1 is complete, then ~~we will guarantee~~

We have T is a homeomorphism and this will complete the proof. So let $\langle x_n \rangle$ be any Cauchy sequence in B , so that

$$\|x_n - x_m\|_1 \rightarrow 0 \text{ as } m, n \rightarrow \infty.$$

$$\Rightarrow \|x_n - x_m\| + \|T(x_n - x_m)\| \rightarrow 0 \text{ as } m, n \rightarrow \infty.$$

$$\Rightarrow \|x_n - x_m\| + \|T(x_n) - T(x_m)\| \rightarrow 0 \text{ as } m, n \rightarrow \infty.$$

$$\Rightarrow \|x_n - x_m\| \rightarrow 0 \text{ and } \|T(x_n) - T(x_m)\| \rightarrow 0 \text{ as } m, n \rightarrow \infty.$$

$\Rightarrow \langle x_n \rangle$ is a Cauchy sequence in B and $\langle T(x_n) \rangle$ is a Cauchy sequence in B' . (1)

Since the graph T_G is given to be closed (1) shows that $(x, y) \in T_G$ so that $y = T(x)$.

$$\text{Now } \|x_n - x\|_1 = \|x_n - x\| + \|T(x_n - x)\|$$

$$= \|x_n - x\| + \|T(x_n) - T(x)\|$$

$$= \|x_n - x\| + \|T(x_n) - y\| \quad [\because y = T(x)]$$

$$[\because x_n \rightarrow x \text{ and } T(x_n) \rightarrow y]$$

It follows that the sequence $\langle x_n \rangle$ in B , converges to $x \in B$, and hence B_1 is complete is required.

(Note that B_1 and B are the same sets so that $x \in B \Rightarrow x \in B_1$).

Proved

Theorem: — Let N and N' be normed linear spaces and D a subspace of N . Then a linear transformation $T: D \rightarrow N'$ is closed if and only if its graph T_G is closed.

Proof: — First suppose T is a closed linear transformation. To show that its graph T_G is closed. i.e. T_G contains all its limit points.

Let (x, y) be any limit point of T_G . Then there exists a sequence of points in T_G , $\langle x_n, T(x_n) \rangle$, where $x_n \in D$, converging to (x, y) .

$$\text{But } \langle x_n, T(x_n) \rangle \rightarrow (x, y)$$

$$\Rightarrow \|(x_n, T(x_n)) - (x, y)\| \rightarrow 0$$

$$\Rightarrow \|x_n - x, T(x_n) - y\| \rightarrow 0$$

$$\Rightarrow \|x_n - x\| + \|T(x_n) - y\| \rightarrow 0$$

$$\Rightarrow \|x_n - x\| \rightarrow 0 \text{ and } \|T(x_n) - y\| \rightarrow 0$$

$$\Rightarrow x_n \rightarrow x \text{ and } T(x_n) \rightarrow y$$

$\Rightarrow x \in D$ and $T(x) = y$ ($\because T$ is closed)
by definition of graph.)

$\Rightarrow (x, y) \in T_G$ by definition of graph.

Thus we have shown that every limit point of T_G is in T_G and so T_G is closed.

Conversely let the graph T_G of T be closed.

TO show that T is a closed linear transformation.

let $x_n \in D$, $x_n \rightarrow x$ and $T(x_n) \rightarrow y$.

Then it is easy to see that (x, y) is an adherent point of T_G so that $(x, y) \in \overline{T_G}$. But $\overline{T_G} = T_G$. Since T_G is given to be closed. Hence $(x, y) \in T_G$ and so by definition of T_G , we have $x \in D$ and $y = T(x)$.

Therefore T is a closed linear transformation.

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